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A non-linear equation incorporating damping and dispersion

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The steady-state solution of the non-linear equation

$$h_t + hh_x + h_{xxx} = \delta h_{xx}$$

with both damping and dispersion is examined in the phase plane. For small damping an averaging technique is used to obtain an oscillatory asymptotic solution. This solution becomes invalid as the period of the oscillation approaches infinity, and is matched to a straightforward expansion solution. The results obtained are compared with a numerical integration of the equation.

1. Introduction

During a study of the propagation of waves on liquid-filled elastic tubes (Johnson 1969), it was found that a particular limit of the problem led to a Korteweg & de Vries (1895) equation with damping:

$$h_t + hh_z + h_{zzz} = \delta h_{zz}, \tag{1}$$

where $h(z, t)$ is proportional to the radial perturbation of the tube wall, and z and t are the characteristic and time variables respectively. The equation (1) was valid in the far-field of an initially linear (small amplitude) near-field solution. This equation is the simplest form of wave equation in which non-linearity (hh_z), dispersion (h_{zzz}) and damping (δh_{zz}) all occur.
Examination of the steady-state form of (1) (especially in the phase plane) showed that the radial profile $h(z - C_0 t)$ was very similar to the observed surface profiles of bores. Indeed, as the damping parameter (δ) is varied, the solution is altered from a monotonic profile to an oscillatory one headed by a near-solitary wave. For the bore, Benjamin & Lighthill (1954) showed that if only some of the classical energy loss occurred at the bore, then the excess could be carried away by a stationary wave train. In fact they showed that the waves could be of the well-known 'cnoidal' form. Chester (1966) also indicated that in some sense a perturbation of Poiseuille flow led to monotonic and oscillatory surface profiles.

The steady-state version of (1) has been suggested by Grad & Hu (1967) to describe the weak shock profile in plasmas, and they discuss the solution in the phase plane.

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In the present paper a phase-plane analysis similar to that of Grad & Hu is described and, for small damping ($\delta \ll 1$), asymptotic solutions are obtained to the steady-state equation by a method due to Kuzmak (1959).

2. Method of analysis

The steady-state form of equation (1) is obtained by considering waves travelling at a uniform speed, so that

$$v(T) = \frac{1}{h_\infty} h(T), \quad T = \frac{z - \frac{1}{2}h_\infty t}{2(2/h_\infty)^{\frac{1}{2}}}, \quad h(-\infty) = h_\infty. \tag{2}$$

Also assuming that the upstream and downstream conditions remain undisturbed

$$v \to 0, \quad T \to \infty; \quad v \to 1, \quad T \to -\infty$$

and hence
$$\frac{d^2v}{dT^2} + 4v(v-1) = \epsilon \frac{dv}{dT}, \quad \epsilon = 2\delta\left(\frac{2}{h_\infty}\right)^{\frac{1}{2}}. \tag{3}$$

It is in the form (3) that we shall be studying the equation.

It is well known that a linear oscillator with a small non-linear term can be studied by the method of averages (e.g. Bogoliubov & Mitropolsky 1961). An asymptotic solution as an expansion in ϵ is obtained by ensuring that successive terms are periodic.

For no damping ($\epsilon = 0$), (3) is a non-linear equation with solutions which perform periodic oscillations—the so-called cnoidal waves of water wave theory. For small damping ($\epsilon \ll 1$), similar application of averaging and periodicity conditions are possible as explained by Kuzmak (1959), but the analysis is considerably more complicated (usually resulting in the integration of Jacobian elliptic functions). The solutions of the damped equation extend over a large period range from the single solitary wave (infinite period) to the zero amplitude zero period cnoidal wave. Since averaging techniques rely on averaging over a large number of wavelengths such methods break down as the solitary wave is approached. It is then necessary to match this solution to a straightforward perturbation of the solitary wave.

3. Phase-plane analysis

Equation (3) can be studied in the phase plane by introducing $w = dv/dT$ so that

$$\frac{dw}{dv} = \frac{\epsilon w - 4v(v-1)}{w}. \tag{4}$$

In the plane (w, v), there are two singular points,

$$w = 0, \quad v = 0, \text{ a saddle point;}$$

$$w = 0, \quad v = 1, \begin{cases} \text{a stable node, } \epsilon \geqslant 4, \\ \text{a stable spiral point, } \epsilon < 4. \end{cases}$$

A sketch of the integral curves for (4) with $4 > \epsilon \geqslant 0$ is given in figure 1. The special case $\epsilon = 0$ is the solitary wave, and clearly any other permitted value of ϵ causes the integral curve to spiral towards $v = 1$. In the physical plane (v, T) this corresponds to a profile which starts as a solitary wave and slowly deviates from it, eventually oscillating about $v = 1$ (with ever decreasing period and amplitude).

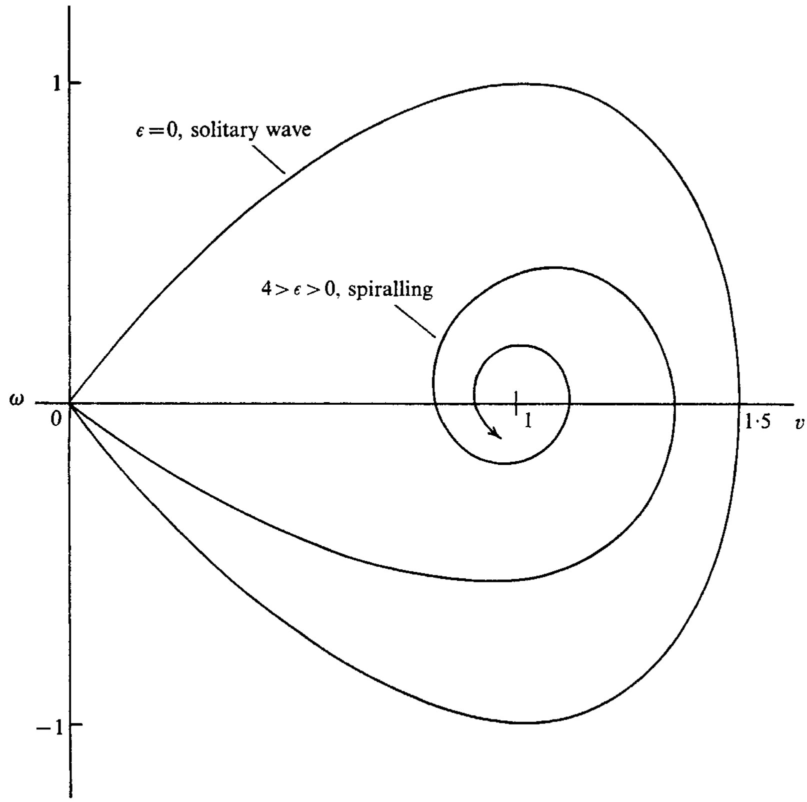

FIGURE 1. Phase plane ($\omega = dv/dT, v$) with $4 > \epsilon \geqslant 0$.

For large ϵ, we write $\epsilon w = w_1$ and (4) becomes

$$\frac{1}{\epsilon^2}\frac{dw_1}{dv} = \frac{w_1 - 4v(v-1)}{w_1}.$$

Consequently the integral curves now become as shown in figure 2. The special case shown of $\epsilon = \infty$ is the Taylor shock profile. As ϵ decreases from infinity, the integral curve deviates from the Taylor profile but still remains monotonic. When $\epsilon < 4$, the curve spirals to $v = 1$ as before. Physically the solution for $v(T)$ remains a monotonically increasing function from upstream to downstream conditions until $\epsilon < 4$, when the oscillatory motion about $v = 1$ occurs.

As mentioned earlier, these results were obtained by Grad & Hu (1967). A numerical integration of (3) confirms this phase-plane analysis, the results of which are shown in figure 3.

The figure shows on the same axes the two basically different types of solution which have been indicated. On the graph are a near-Taylor shock profile ($\epsilon = 4{\cdot}62$); a profile which is just oscillatory ($\epsilon = 3{\cdot}28$) (i.e. the curve just exceeds unity and further oscillations are too small to be plotted); a much larger amplitude oscillation ($\epsilon = 0{\cdot}1$); and finally, for comparison, the solitary wave ($\epsilon = 0$). Clearly the nature of these solutions agrees with the phase-plane analysis.

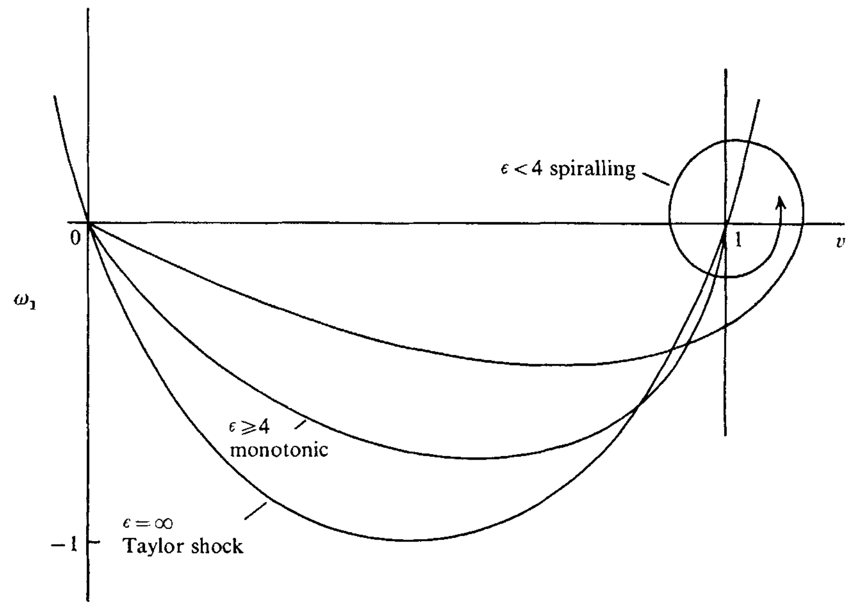

FIGURE 2. Phase plane ($\omega_1 = \epsilon(dv/dT), v$) with $\epsilon \geqslant 4$ and $\epsilon < 4$.

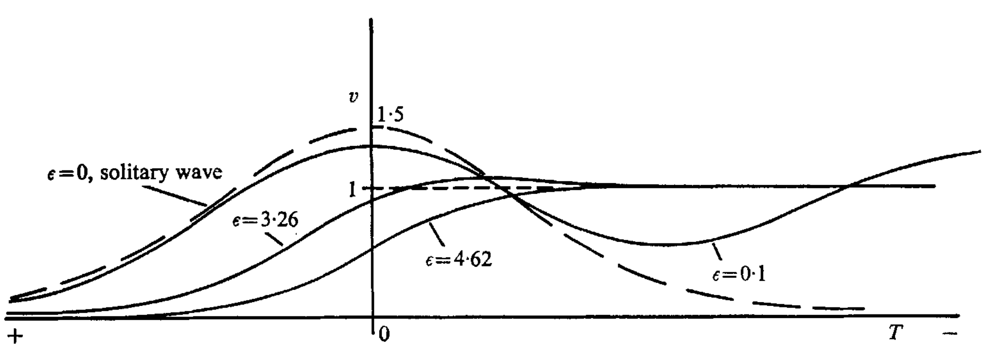

FIGURE 3. Various integrations of (3) showing transition from monotonic to oscillatory profiles.

4. Oscillatory asymptotic analysis

Figure 1 shows that for small damping the solution starts very near to a solitary wave and steadily departs from it, causing the profile to oscillate about $v = 1$. It is evident that the small damping causes the period of the oscillation to slowly change from infinite (the solitary wave) to zero (at $v = 1$).

This suggests that a solution to (3), for small damping, can be found by allowing the amplitude and period of the undamped solution to slowly vary. This idea together with the insistence on periodicity is the approach used by Kuzmak (1959), and indeed any method involving averaged periodic solutions.

To utilize this method we must find the undamped solution to equation (3) (Lamb 1956). Thus considering the slightly more general equation without damping,

$$d^2v/dT^2 + 4v(v-1) = A,$$

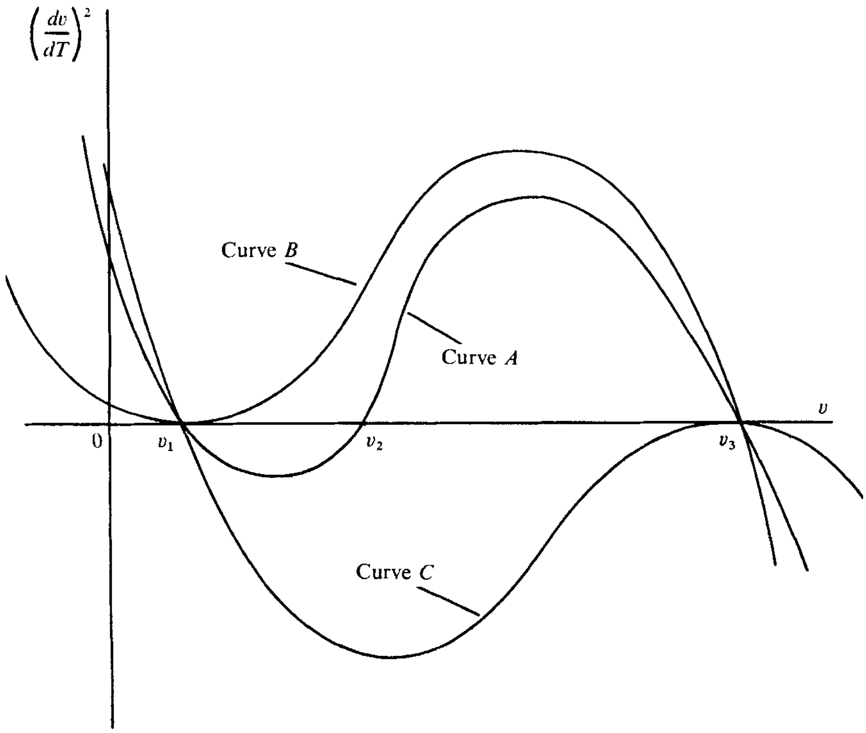

FIGURE 4. Sketch of the cubic expression for the general (undamped) cnoidal wave.

where A is a constant. Integrating once,

$$\begin{aligned}\tfrac{3}{8}(dv/dT)^2 &= -v^3 + \tfrac{3}{2}v^2 + \tfrac{3}{4}Av + B \\ &= (v-v_1)(v-v_2)(v_3-v),\end{aligned}$$

where $v_1 < v_2 < v_3$. The general form of this cubic expression is sketched in figure 4 (curve A).

The only real solution of the equation occurs for $(dv/dT)^2 \geqslant 0$, and thus the solution is either at $v = v_1$ or a non-linear oscillation between v_2 and v_3 Two special cases of curve A are $v_2 \to v_1$ (curve B) giving the solitary wave, and $v_2 \to v_3$ (curve C) giving a discontinuity between v_1 and v_3 (the hydraulic jump).

The solution can be written in terms of a Jacobian elliptic function cn $(u;\, \nu)$ as

$$v = v_2 + (v_3 - v_2)\,\mathrm{cn}^2\,[T[\tfrac{2}{3}(v_3 - v_1)]^{\frac{1}{2}};\, \nu], \tag{5}$$

where $\nu = (v_3 - v_2)/(v_3 - v_1)$. See Abramowitz & Stegun (1965). In the case of $A = B = 0$, then $\nu = 1$ and the solution becomes $v = \frac{3}{2}\,\mathrm{sech}^2\,T$, the solitary wave.

Now introducing a slow time scale

$$\tau = \epsilon T \tag{6}$$

a two parameter (T, τ) expansion is sought,

$$v = v_0(T, \tau) + \epsilon v_1(T, \tau) + \dots. \tag{7}$$

The form of (5) indicates that

$$v_0 = a(\tau) + b(\tau)\,\mathrm{cn}^2\,[T\,.\,\alpha(\tau);\, \nu(\tau)], \tag{8}$$

where $\alpha(\tau) = (2b/3\nu)^{\frac{1}{2}}$, so that (8) describes the slowly-varying nature of the solution. Using (6) we have

$$\frac{d}{dT} \equiv \frac{\partial}{\partial T} + \epsilon\frac{\partial}{\partial \tau}, \quad \frac{d^2}{dT^2} \equiv \frac{\partial^2}{\partial T^2} + 2\epsilon\frac{\partial^2}{\partial T\,\partial \tau} + \epsilon^2\frac{\partial^2}{\partial \tau^2}$$

which together with (7) is substituted into (3) giving

$$O(1)\colon \frac{\partial^2 v_0}{\partial T^2} + 4v_0(v_0 - 1) = 0, \tag{9}$$

$$O(\epsilon)\colon \frac{\partial^2 v_1}{\partial T^2} + 4v_1(2v_0 - 1) = \frac{\partial v_0}{\partial T} - 2\frac{\partial^2 v_0}{\partial \tau\,\partial \tau}. \tag{10}$$

Putting (8) into (9) and using

$$-\,\mathrm{dn}^2 + 1 - \nu = -\nu\,\mathrm{cn}^2 = \nu(\mathrm{sn}^2 - 1)$$

gives

$$(4b^2/3\nu)[1 - \nu + 2(2\nu - 1)\,\mathrm{cn}^2 - 3\nu\,\mathrm{cn}^4] + 4(a + b\,\mathrm{cn}^2)^2 - 4(a + b\,\mathrm{cn}^2) = 0,$$

and equating the coefficients of cn^2, cn^4,

$$a = \tfrac{1}{2} - \tfrac{1}{3}(b/\nu)(2\nu - 1), \quad b = \tfrac{3}{2}\nu(\nu^2 - \nu + 1)^{-\frac{1}{2}}. \tag{11}$$

The third equation is identically satisfied due to the choice of $\alpha(\tau)$.

We now need one more relation to enable a, b and ν to be defined. This is obtained from the condition that $v_1(T, \tau)$ be periodic in T. To simplify (10), we put

$$v_1 = v_{0_T}\cdot f(T, \tau) \tag{12}$$

giving

$$f_T = \frac{1}{v_{0_T}^2}\int_{b_0}^{T} (v_{0_T}^2 - 2_{0_T} v_{0_{T\tau}})\,dT, \tag{13}$$

where b_0 is arbitrary. Now v_1 is periodic if f_T is, thus

$$\int_{a-\frac{1}{2}T_p}^{a+\frac{1}{2}T_p} (v_{0_T}^2 - 2v_{0_T} v_{0_{T\tau}})\,dT = 0,$$

where T_p is the period of f.

Hence

$$C_1 e^{\tau} = \int_{a-\frac{1}{2}T_p}^{a+\frac{1}{2}T_p} (v_{0_T})^2\,dT, \tag{14}$$

where C_1 is constant. Equation (14) can be rewritten, by averaging over a period, as

$$C_2 e^{\tau} = b^2\left(\frac{2b}{3\nu}\right)^{\frac{1}{2}}\int_0^{K(\nu)} \mathrm{cn}^2 y\,\mathrm{sn}^2 y\,\mathrm{dn}^2 y\,dy = b^2\left(\frac{2b}{3\nu}\right)^{\frac{1}{2}} L(\nu) \tag{15}$$

($K(\nu)$ defined below). Using the definition of the Jacobian elliptic functions, $L(\nu)$ becomes

$$L(\nu) = \int_0^{\frac{1}{2}\pi} \cos^2\phi \sin^2\phi (1-\nu\sin^2\phi)^{\frac{1}{2}}\, d\phi. \tag{16}$$

With the well-known relations

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt \quad (\mathscr{R}c > \mathscr{R}b > 0),$$

and

$$\left.\begin{aligned} F(\tfrac{1}{2},\tfrac{1}{2},1;\nu) &= \tfrac{1}{2}\pi K(\nu) \\ F(\tfrac{1}{2},-\tfrac{1}{2},1;\nu) &= \tfrac{1}{2}\pi E(\nu) \end{aligned}\right\} \begin{aligned} &\text{the complete elliptic integrals of} \\ &\text{the 1st and 2nd kind respectively} \end{aligned}$$

relation (16) can be written as

$$\nu^2 L(\nu) \propto (\nu^2-\nu+1)E(\nu) - \tfrac{1}{2}(1-\nu)(2-\nu)K(\nu), \tag{17}$$

which together with

$$e^{\tau} \propto \nu^2(\nu^2-\nu+1)^{-\frac{5}{4}}L(\nu)$$

becomes the third relation defining $a(\tau), b(\tau), \nu(\tau)$. The constant of proportionality is to be found by putting $\nu = 1$ at $T = T_0$, so that

$$e^{\tau-\tau_0} = (\nu^2-\nu+1)^{-\frac{5}{4}}[(\nu^2-\nu+1)E(\nu) - \tfrac{1}{2}(1-\nu)(2-\nu)K(\nu)], \tag{18}$$

where T_0 is the apparent position where the damping is 'switched-on' for the oscillatory analysis. This can only be fixed by matching.

The final solution given by (7), (12) and (13) is

$$v = v_0 + \epsilon v_{0T}\int_{a_0}^{T} \frac{1}{v_{0T}^2}\left[\int_{b_0}^{T} (v_{0T}^2 - 2v_{0T}v_{0T\tau})\,dT\right]dT + \dots, \tag{19}$$

where a_0 is arbitrary. Kuzmak points out that a solution like (19) is only valid provided that $T_p < \infty$ (see (14)) i.e. if v_0 is truly oscillatory. Consequently (19) becomes invalid as $\nu \to 1$, since v_0 becomes non-periodic (see (5)).

5. Perturbation of the solitary wave

Since (19) becomes invalid, it is necessary to obtain a non-oscillatory solution of (3) by performing a straightforward expansion in ϵ. In fact we shall see that this amounts to a perturbation of the solitary wave.

For convenience (3) is integrated once,

$$\frac{1}{2}\left(\frac{dv}{dT}\right)^2 = \epsilon\int_{\infty}^{T}\left(\frac{dv}{dT}\right)^2 dT - \tfrac{4}{3}v^3 + 2v^2, \tag{20}$$

remembering that $v = dv/dT = 0$ at $T = +\infty$, so that the solution of (20) will be monotonic as $T \to +\infty$. Clearly (15), and hence the solution (19), is valid as $T \to -\infty$ describing the oscillatory nature of the solution. A straightforward expansion is

$$v = V_0(T) + \epsilon V_1(T) + \dots, \tag{21}$$

which gives

$$O(1)\colon \frac{1}{2}\left(\frac{dV_0}{dT}\right)^2 = 2V_0^2 - \tfrac{4}{3}V_0^3, \tag{22}$$

$$O(\epsilon)\colon \frac{dV_0}{dT}\frac{dV_1}{dT} = \int_{\infty}^{T}\left(\frac{dV_0}{dT}\right)^2 dT + 4V_0V_1(1-V_0). \tag{23}$$

The solution to (22) is the solitary wave,

$$V_0 = \tfrac{3}{2}\operatorname{sech}^2 T, \tag{24}$$

where the peak is fixed at $T = 0$. Substituting this into (23) and integrating the equation gives

$$V_1 = \tfrac{2}{5}\operatorname{sech}^2 T(\tanh T - 1) + \tfrac{1}{20}\tanh T(> \tanh T - 6) \\ - \frac{\tanh T}{10(1+\tanh T)} + \tfrac{3}{4}T\tanh T\operatorname{sech}^2 T \tag{25}$$

where the condition $dV_1/dT|_{T=0} = 0$ ensures that the peak occurs at $T = 0$ to $O(\epsilon^2)$. It is clear that the expansion (21) becomes invalid as

$$\tanh T \to -1, \quad T \to -\infty,$$

since the expression (25) then approaches infinity.†

6. Matching procedure

The solution given by (7) becomes invalid as $T \to \infty$, since the period of the wave approaches infinity and thus the averaging assumption is invalidated. It is possible, however, to expand the oscillatory solution as $\nu \to 1$ (i.e. becomes non-oscillatory), so that by suitable choice of a_0 and b_0 (in (19)) it agrees term by term with the perturbation of the solitary wave ((24) and (25)). This is therefore a matching procedure for these two very dissimilar expansions. The method outlined here does not follow the usual technique of matched asymptotic expansions (Van Dyke 1964), where the breakdown of one expansion leads to a new scaling of the equation and consequently a new asymptotic expansion. Matching of two such expansions then follows directly.

Thus we expand the oscillatory solution (8) and (9) for $\nu \to 1$, $T \to T_0$ and in particular consider the region where

$$T - T_0 = O(1), \quad \nu = 1 - \delta(\epsilon).\nu_1, \quad \delta(\epsilon) \ll 1, \quad \nu_1 > 0; \tag{26}$$

then (18) gives

$$T \sim T_0 - \tfrac{33}{32}\nu_1^2, \quad \delta^2 \ln\delta = -\epsilon, \tag{27}$$

so that $T < T_0$ for $\nu < 1$.

Now

$$v_{0T} = -2\alpha b \operatorname{cn}(\alpha T)\operatorname{sn}(\alpha T)\operatorname{dn}(\alpha T) \sim -3\operatorname{sech}^2 T\tanh T. \tag{28}$$

Thus

$$\int_{b_0}^{T}(v_{0T}^2 - 2v_{0T}v_{0T\tau})\,dT \sim 9[B + \tfrac{1}{3}\tanh^3 T - \tfrac{1}{5}\tanh^5 T], \; v_{0T\tau} = O(\delta),$$

where

$$B = \tfrac{1}{5}\tanh^5 b_0 - \tfrac{1}{3}\tanh^3 b_0. \tag{29}$$

Also

$$\int_{a_0}^{T}\frac{1}{v_{0T}^2}\left[\int_{b_0}^{T}(v_{0T}^2 - 2v_{0T}v_{0T\tau})\,dT\right]dT \sim B\left[\tfrac{15}{8}T - \frac{1}{\tanh T} + \frac{\tanh T}{\operatorname{sech}^2 T} \right. \\ \left. + \frac{1}{8}\frac{\tanh T}{\operatorname{sech}^4 T} + \frac{1}{8}\frac{\tanh^3 T}{\operatorname{sech}^4 T}\right] + \tfrac{1}{12}\cosh^4 T - \tfrac{1}{20}\sinh^4 T + A, \tag{30}$$

where

$$A = -B\left[\tfrac{15}{8}a_0 - \frac{1}{\tanh a_0} + \frac{\tanh a_0}{\operatorname{sech}^2 a_0} + \frac{1}{8}\frac{\tanh a_0}{\operatorname{sech}^4 a_0} + \frac{1}{8}\frac{\tanh^3 a_0}{\operatorname{sech}^4 a_0}\right] \\ - \tfrac{1}{12}\cosh^4 a_0 + \tfrac{1}{20}\sinh^4 a_0. \tag{31}$$

† Note that the expansion also becomes invalid as $T \to +\infty$, but it is elementary to show that the solution here can be matched directly to the asymptotic exponential behaviour of the solitary wave.

Finally $$v_0 = \tfrac{3}{2}\,\mathrm{sech}^2 T + O(\delta^2). \tag{32}$$

Thus the expression (19) as $\nu \to 1$ can be written as

$$v \sim \tfrac{3}{2}\,\mathrm{sech}^2 T + \epsilon\Big[3B + (-\tfrac{3}{20} - 3A)\tanh T - \tfrac{45}{8}B\tanh^2 T + (-\tfrac{3}{20} + 3A)\tanh^3 T - \frac{1}{10}\frac{\tanh T}{\mathrm{sech}^2 T} - \frac{3B}{4}\frac{\tanh^2 T}{\mathrm{sech}^2 T} - \tfrac{45}{8}BT\tanh T\,\mathrm{sech}^2 T\Big]. \tag{33}$$

Expression (33) matches exactly with (24) and (25) if

$$A = -\tfrac{1}{12}, \quad B = -\tfrac{2}{15}. \tag{34}$$

Consequently from (29), $b_0 = +\infty$, and the one finite real solution of (31) gives $a_0 = -0{\cdot}529$ by a numerical computation.

A sketch of the regions of validity of the asymptotic expansions is given in figure 5.

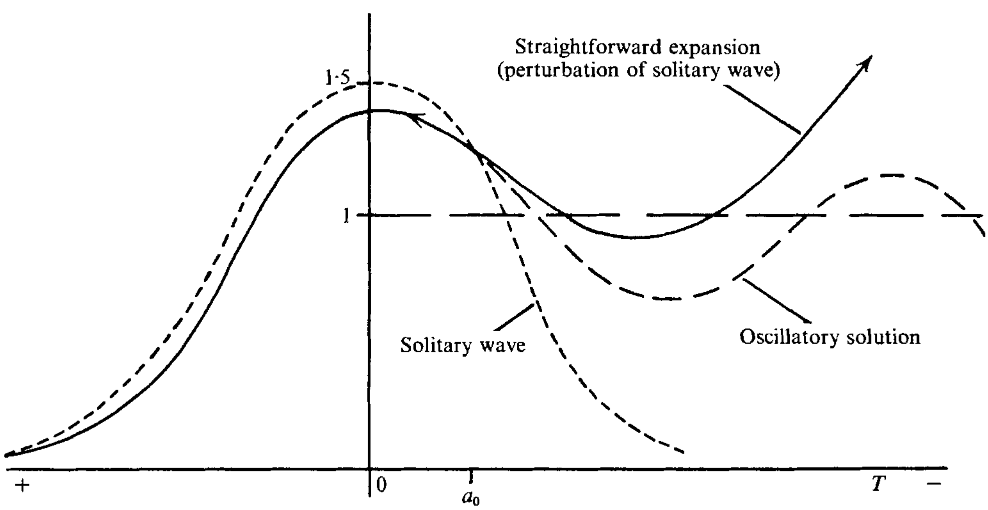

FIGURE 5. A sketch showing the two asymptotic expansions, and the solitary wave. Note that both expansions cut the solitary wave at $T = a_0$ (to $O(\delta^2)$).

The matching of the expressions (33) and (24), (25) did not entail finding ν_1 (i.e. T_0). Since (32) was correct to $O(\delta^2)$, and ν_1 did not appear, then to the approximation used in this paper it is impossible to define T_0 (the apparent point where viscosity is 'switched-on' for the oscillatory analysis). The only statement that can be made is

$$T - T_0 = O(1) \quad (\nu \to 1)$$

and since the expression (24), (25) is only valid for $T \leqslant O(1)$, $T_0 = O(1)$.

This result is not really unexpected. For the limit of small damping, the rate of dissipation of energy through the first wave is also small, and consequently the exact position of T_0 becomes irrelevant to $O(\epsilon)$. Thus to $O(1)$, $e^{-\tau_0} = 1$. It is now possible to compute $a(\tau)$, $b(\tau)$, and $\nu(\tau)$, and in figure 6 the computed curves of a, $a+b$ and ν are given as functions of e^{τ}. From (6) it is clear that $(a+b)$ and a are the upper and lower envelopes respectively.

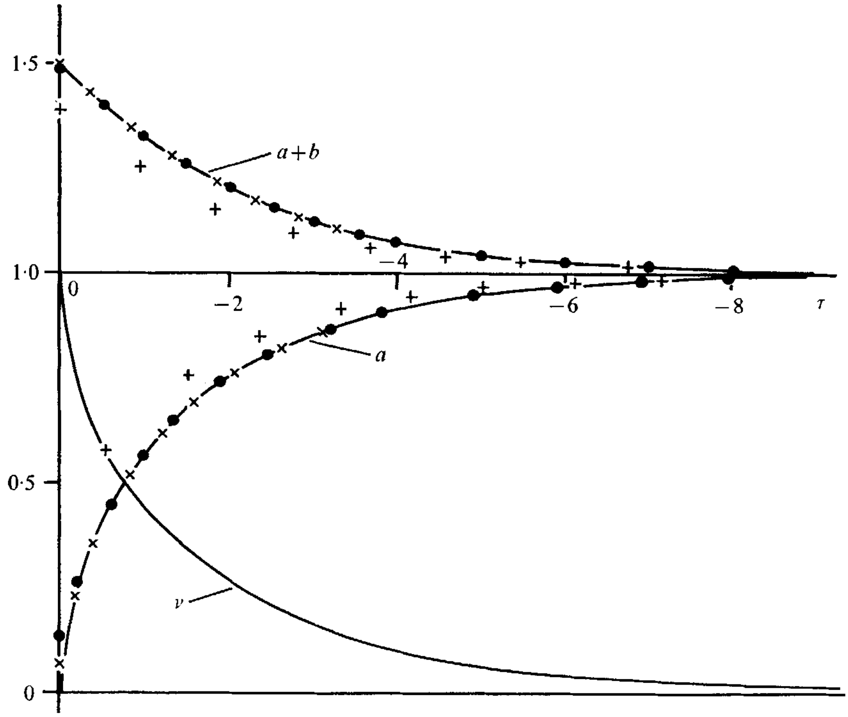

FIGURE 6. Computed curves for upper and lower envelopes and $\nu(\tau)$. Peak envelopes obtained by numerical integration for: $+$, $\epsilon = 0{\cdot}1$; ●, $\epsilon = 0{\cdot}01$; $\times$, $\epsilon = 0{\cdot}001$ plotted for comparison. —, theory.

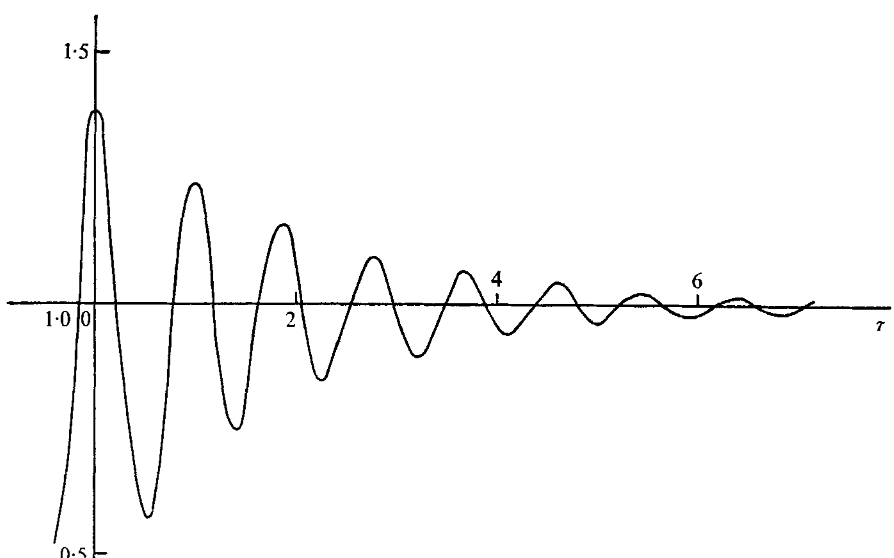

FIGURE 7. Results of the numerical integration of (3) for $\epsilon = 0{\cdot}1$, as a function of τ.

Also plotted on figure 6 are some peaks obtained by numerical integration of equation (3) using a predictor-corrector technique (for $\epsilon = 0{\cdot}1$, $0{\cdot}01$ and $0{\cdot}001$). It is seen that the agreement for the two smaller values of ϵ is excellent, and for $\epsilon = 0{\cdot}1$, fair. These results thus show the validity of the theory for $\epsilon \to 0$. The form of the oscillatory solution is shown in figure 7, where the numerical results for $\epsilon = 0{\cdot}1$ are plotted against τ.

7. The monotonic profile

To complete the analyses, we will perturb the monotonic profile. It was seen in § 3 that equation (3) can represent the Taylor shock profile as $\epsilon \to \infty$. Rescaling (3) by using

$$T = \epsilon\mathscr{T}$$

then

$$\frac{1}{\epsilon^2}\frac{d^2v}{d\mathscr{T}^2} + 4v(v-1) = \frac{dv}{d\mathscr{T}}.$$

It is clear that as $\epsilon \to \infty$, the equation becomes the usual form for a Taylor shock between the two levels $v = 0$ and $v = 1$.

A straightforward asymptotic expansion

$$v = \mathscr{V}_0(\mathscr{T}) + \frac{1}{\epsilon^2}\mathscr{V}_1(\mathscr{T}) + \ldots,$$

gives

$$O(1)\colon 4\mathscr{V}_0(\mathscr{V}_0 - 1) = d\mathscr{V}_0/d\mathscr{T}$$

$$O(1/\epsilon^2)\colon d^2\mathscr{V}_0/d\mathscr{T}^2 + 4\mathscr{V}_1(\mathscr{V}_0 2 - 1) = d\mathscr{V}_1/d\mathscr{T}.$$

Thus it is easy to show that

$$\mathscr{V}_0 = \tfrac{1}{2}[1 + \tanh(-2\mathscr{T})]$$

and

$$\mathscr{V}_1 = \operatorname{sech}^2(-2\mathscr{T})\,[A - \ln\{\operatorname{sech}^2(1 - 2\mathscr{T})\}].$$

Fixing the midpoint of the profile at $\mathscr{T} = 0$ gives $A = 0$, and thus the near-Taylor shock profile becomes

$$v = \tfrac{1}{2}[1 + \tanh(-2\mathscr{T})] - (1/\epsilon^2)\operatorname{sech}^3(-2\mathscr{T})\ln[\operatorname{sech}^2(1 - 2\mathscr{T})] + \ldots.$$

8. Discussion

The slowly-varying (oscillatory) analysis has been successfully matched to a straightforward asymptotic expansion in ϵ. The method used (by approximating to $O(\delta^2)$) leaves us with an arbitrariness in the oscillatory analysis—the point (T_0) where the damping effect appears to begin is not specified other than to $O(1)$. Including the next approximation (which would be exceedingly tedious) should enable this position to be defined. To obtain the envelope curves in the oscillatory region of the solution, $e^{-\tau_0}$ has been approximated by unity and the results are given in figure 6. As already seen the numerical integration confirms the predicted envelope curves.

This work has shown that the solution to (3) is (for small damping) a cnoidal wave with varying period passing through all the possible wave types and culminating in the solitary wave. Such a wave profile is exactly what has been observed in the undular bore. The measurements of Sandover & Taylor (1962) in fact show that the leading wave is very close indeed to the solitary wave. Thus we can tentatively suggest (3) as a model equation describing the undular bore (and Grad & Hu (1967) suggested it for the weak shock in plasmas). But it is also observed that some bores do not have a stationary wave train behind them—the hydraulic jump—and as explained in the phase plane this occurs for $\epsilon \geqslant 4$, when the profile is monotonic. However, equation (3) has not been shown to describe the undular bore in any true asymptotic limit.

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